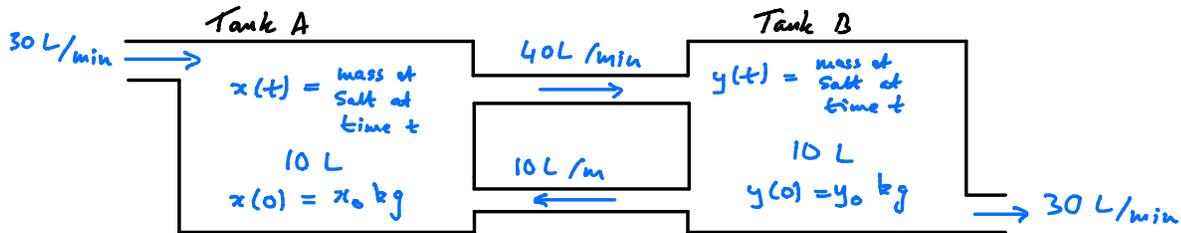


Linear Systems of Differential Equations

Motivating Example

Physical System : Two tanks A and B of salty water. They are connected and fresh water is pumped in as follows



Q, : $x(t), y(t) = ?$

$$x'(t) = \text{Input Rate} - \text{Output Rate}$$

$$= \frac{10}{10} y(t) - \frac{40}{10} x(t) = -4x(t) + y(t)$$

$$y'(t) = \text{Input Rate} - \text{Output Rate}$$

$$= \frac{40}{10} x(t) - \frac{10}{10} y(t) - \frac{30}{10} y(t) = 4x(t) - 4y(t)$$

Conclusion : Need to find $x(t), y(t)$ such that

$$x' = -4x + y$$

$$y' = 4x - 4y \quad (\Rightarrow 4x = y' + 4y)$$

Approach 1 :

$$y' = 4x - 4y \Rightarrow y'' = 4x' - 4y' \Rightarrow y'' = 4(-4x + y) - 4y'$$

$$\Rightarrow y'' = 4(-y' - 4y + y) - 4y'$$

$$\Rightarrow y'' + 8y' + 12y = 0$$

$$\leadsto r^2 + 8r + 12 = 0 \Rightarrow (r+2)(r+6) = 0 \Rightarrow r = -2, -6$$

$$\Rightarrow y(t) = c_1 e^{-2t} + c_2 e^{-6t}$$

$$\Rightarrow y'(t) = -2c_1 e^{-2t} - 6c_2 e^{-6t}$$

$$\begin{aligned} \Rightarrow x(t) &= \frac{1}{4} y'(t) + y(t) = \frac{1}{4} (-2c_1 e^{-2t} - 6c_2 e^{-6t}) \\ &\quad + c_1 e^{-2t} + c_2 e^{-6t} \\ &= \frac{1}{2} c_1 e^{-2t} - \frac{1}{2} c_2 e^{-6t} \end{aligned}$$

$$\begin{aligned} \Rightarrow x(0) &= \frac{1}{2} c_1 - \frac{1}{2} c_2 = x_0 \\ y(0) &= c_1 + c_2 = y_0 \end{aligned} \quad \left. \vphantom{\begin{aligned} \Rightarrow x(0) \\ y(0) \end{aligned}} \right\} \begin{aligned} c_1 &= \frac{2x_0 + y_0}{2} \\ c_2 &= \frac{y_0 - 2x_0}{2} \end{aligned}$$

$$\Rightarrow x(t) = \frac{1}{2} \left(\frac{2x_0 + y_0}{2} \right) e^{-2t} - \frac{1}{2} \left(\frac{y_0 - 2x_0}{2} \right) e^{-6t}$$

$$y(t) = \left(\frac{2x_0 + y_0}{2} \right) e^{-2t} + \left(\frac{y_0 - 2x_0}{2} \right) e^{-6t}$$

Approach 2 Use Linear algebra.

Observation :

$$\begin{aligned} x'(t) &= -4x(t) + y(t) \\ y'(t) &= 4x(t) - 4y(t) \end{aligned} \Rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Imagine λ is an eigenvalue of $\begin{pmatrix} -4 & 1 \\ 4 & -4 \end{pmatrix}$ and \underline{v} in \mathbb{R}^2

is an eigenvector.

$$\text{Consider } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \underline{v}$$

$$\Rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \lambda e^{\lambda t} \underline{v} = e^{\lambda t} \lambda \underline{v} = e^{\lambda t} A \underline{v} = A (e^{\lambda t} \underline{v}) = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Conclusion : $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \underline{v}$ solution to $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\det(A - xI_2) = (-4 - x)(-4 - x) - 4 = x^2 + 8x + 12 = 0$$

$$\Rightarrow x = -2, -6$$

$$\text{Nul}(A + 2I_2) = \text{Nul} \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} = \text{Span} \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

$$\text{Nul}(A + 6I_2) = \text{Nul} \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = \text{Span} \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right)$$

$$\Rightarrow e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, e^{-6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ are potential values of } \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

$$\Rightarrow d_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + d_2 e^{-6t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ give potential solutions}$$

$$\begin{pmatrix} d_1 e^{-2t} + d_2 e^{-6t} \\ 2d_1 e^{-2t} - 2d_2 e^{-6t} \end{pmatrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \Rightarrow \begin{pmatrix} d_1 + d_2 \\ 2d_1 - 2d_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & x_0 \\ 2 & -2 & y_0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & x_0 \\ 0 & -4 & y_0 - 2x_0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & \frac{2x_0 + y_0}{4} \\ 0 & 1 & \frac{2x_0 - y_0}{4} \end{array} \right)$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{2x_0 + y_0}{4} e^{-2t} + \frac{2x_0 - y_0}{4} e^{-6t} \\ \frac{2x_0 + y_0}{2} e^{-2t} - \frac{2x_0 - y_0}{2} e^{-6t} \end{pmatrix}$$

Awesome!

Let's look at the general picture.

Cool Example $n = m$, $A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -p_0(t) & 1 \\ -p_0(t) & -p_1(t) & \dots & -p_{n-1}(t) & 0 \end{pmatrix}$, $f(t) = \begin{pmatrix} 0 \\ \vdots \\ g(t) \end{pmatrix}$

$$\underline{x}'(t) = A(t) \underline{x}(t) + f(t)$$

$$\Leftrightarrow x_1'(t) = x_2(t)$$

$$x_2'(t) = x_3(t)$$

⋮

$$x_{n-1}'(t) = x_n(t)$$

$$x_n'(t) = -p_0(t)x_1(t) - p_1(t)x_2(t) \dots - p_{n-1}(t)x_n(t) + g(t)$$

$$\Leftrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_1'(t) \\ x_1''(t) \\ \vdots \\ x_1^{(n-1)}(t) \end{pmatrix} \quad \text{and}$$

Linear n^{th} order differential equation

$$x_1^{(n)}(t) + p_{n-1}(t)x_1^{(n-1)}(t) + \dots + p_1(t)x_1'(t) + p_0(t)x_1(t) = g(t)$$

Conclusion:

$$\text{Solving } \underline{x}'(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -p_0(t) & 1 \\ -p_0(t) & -p_1(t) & \dots & -p_{n-1}(t) & 0 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 0 \\ \vdots \\ g(t) \end{pmatrix}$$

is the same as solving

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \dots + p_0(t)y(t) = g(t)$$

Theorem If $A(t)$, $f(t)$ are continuous on open interval I containing t_0 , then for any choice \underline{x}_0 in \mathbb{R}^n , there exists a unique $\underline{x}(t)$ on I such that

each entry continuous

called initial condition

$$1/ \quad \underline{x}'(t) = A(t) \underline{x}(t) + f(t) \quad , \quad 2/ \quad \underline{x}(t_0) = \underline{x}_0$$

Homogeneous Case : $\underline{f}(t) = \underline{0}(t)$ \leftarrow zero function in every coordinate ie. $\underline{x}'(t) = A(t)\underline{x}(t)$

Facts (Homogeneous Case)

- 1/ $\underline{0}(t)$ is a solution to $\underline{x}'(t) = A(t)\underline{x}(t)$
- 2/ $\underline{x}(t), \underline{y}(t)$ solutions $\Rightarrow \underline{x}(t) + \underline{y}(t)$ a solution
- 3/ $\underline{x}(t)$ solution, λ real $\Rightarrow \lambda \underline{x}(t)$ a solution

\Rightarrow $\left\{ \text{Solutions to } \underline{x}'(t) = A(t)\underline{x}(t) \right\} = \text{vector space}$

$T : \left\{ \text{Solutions to } \underline{x}'(t) = A(t)\underline{x}(t) \right\} \longrightarrow \mathbb{R}^n$ Linear, one-to-one, onto
By Theorem

$\underline{x}(t) \longmapsto \underline{x}(t_0)$

$\Rightarrow \dim \left\{ \text{Solutions to } \underline{x}'(t) = A(t)\underline{x}(t) \right\} = n$

Need to find n L.I. solutions.

Q: How can we easily check if n solutions $\underline{x}_1, \dots, \underline{x}_n$ are L.I.?

Recall: $\{\underline{x}_1, \dots, \underline{x}_n\}$ Linearly Dependent if there

exist c_1, \dots, c_n real numbers, not all zero such that

$$c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \dots + c_n \underline{x}_n(t) = \underline{0}(t) \quad (\text{for all } t)$$

$= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

Definition For n, \mathbb{R}^n -valued functions $\underline{x}_1, \dots, \underline{x}_n$,

the Wronskian of $\{\underline{x}_1, \dots, \underline{x}_n\}$ is the \mathbb{R} -valued function

$$W[\underline{x}_1, \dots, \underline{x}_n](t) = \det \begin{pmatrix} \underline{x}_1(t) & \dots & \underline{x}_n(t) \end{pmatrix}$$

Example $\underline{x}_1(t) = \begin{pmatrix} t^2 \\ 0 \\ 0 \end{pmatrix}$, $\underline{x}_2(t) = \begin{pmatrix} \cos(t) \\ e^t \\ 0 \end{pmatrix}$, $\underline{x}_3(t) = \begin{pmatrix} t^2 \\ t^{-1} \\ \cos(t) \end{pmatrix}$

$$\Rightarrow W[\underline{x}_1, \underline{x}_2, \underline{x}_3](t) = \det \begin{pmatrix} t^2 & \cos(t) & t^2 \\ 0 & e^t & t^{-1} \\ 0 & 0 & \cos(t) \end{pmatrix}$$

$$= t^2 e^t \cos(t)$$

Theorem If $\underline{x}_1, \dots, \underline{x}_n$ are solutions to $\underline{x}'(t) = A(t)\underline{x}(t)$

then $\{\underline{x}_1, \dots, \underline{x}_n\}$ L.D. $\Leftrightarrow W[\underline{x}_1, \dots, \underline{x}_n](t_0) = 0$ for some t_0 in I

Proof

(\Rightarrow)

$\{\underline{x}_1, \dots, \underline{x}_n\}$ L.D. \Rightarrow We can find c_1, \dots, c_n , not all zero such that

$$c_1 \underline{x}_1(t) + \dots + c_n \underline{x}_n(t) = \underline{0}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow \{\underline{x}_1(t), \dots, \underline{x}_n(t)\} \subset \mathbb{R}^n$ L.D. for all t in I

$\Rightarrow W[\underline{x}_1, \dots, \underline{x}_n](t) = \det(\underline{x}_1(t) \dots \underline{x}_n(t)) = 0$ for all t in I .

(\Leftarrow) Assume there exists t_0 in I such that $W[\underline{x}_1, \dots, \underline{x}_n](t_0) = 0$

\Rightarrow There exist c_1, \dots, c_n , not all zero such that

Definition If $\{\underline{x}_1, \dots, \underline{x}_n\}$ is a L.I. set of solutions to

$$\underline{x}'(t) = A(t)\underline{x}(t) \text{ we say it is a } \underline{\text{fundamental solution set}}$$

Non-Homogeneous Case : $x'(t) = A(t)x(t) + f(t)$

not $0(t)$

Fact : If \underline{x}_p is a particular solution is

$$\underline{x}'_p(t) = A(t)\underline{x}_p(t) + f(t)$$

general homogeneous solution.

$$\Rightarrow \text{General Solution to } = \underline{x}_p + \underline{x}_h$$

$$x'(t) = A(t)x(t) + f(t)$$

$$\lambda_1 \underline{x}_1 + \dots + \lambda_n \underline{x}_n$$

fundamental solution set